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product of the number of those for the modulus p by that for the modulus q. But since a residue of pq must necessarily be a residue of p and of q separately every pair of consecutive residues for pq will also be a pair of consecutive residues for p and q.

As an illustration of the method of finding consecutive residues for a composite modulus take the modulus $253 = 11.23$. For the modulus 11 we have the consecutive residues 3, 4 arising from 5^2 and 2^2. For the modulus 23 we have the residues 1, 2 arising from 1^2 and 5^2. The congruences $2 + 11k \equiv 5(\text{mod } 23)$ and $5 + 11 \cdot l \equiv 1(\text{mod } 23)$ give $k \equiv 17$, and $l \equiv 8(\text{mod } 23)$. Then $2 + 187$, or 189, and $5 + 88$ or 93 must furnish a pair of consecutive residues for 253. It is found that $189^2 \equiv 48$ and $93^2 \equiv 47$. Since further there are 3 consecutive residues for the modulus 11, and 6 for the modulus 23 there will be 18 for the modulus 253.

In conclusion, a word should be said as to the determination of the possibility of the congruence $x^2 \equiv D(\text{mod } m)$ where D is not restricted to be prime to m. In the first place, it is necessary to consider only the case where the modulus is a power of a prime. For if x can be found such that $x^2 - D$ is divisible by pq, where p and q are relatively prime, then $x^2 - D$ will be divisible by p and q, and conversely. Consider then the modulus p^α; and let $D = Ap^\lambda$ where A is prime to p.

If λ is greater than α the congruence will have the root $x \equiv 0$.

I. Suppose that λ is less than α, and suppose first that A is a residue p of and write $A \equiv y^2(\text{mod } p^\alpha)$. Then if λ is even the congruence is possible and has the root $x \equiv \pm yp^{\lambda/2}$.

II. If A is a residue of p and λ is odd the congruence has no roots. For writing $x^2 \equiv Ap^\lambda + Mp^\alpha$ then x contains p^λ as a factor. It must then contain another factor p. But the right side can contain no such factor.

III. If A is a non-residue of p and λ is even the congruence is impossible. For dividing out p^λ we would have the impossible congruence

$$x'^2 \equiv A(\text{mod } p^{\alpha-2}).$$

IV. If A is a non-residue, and λ is odd then Ap^λ is a non-residue. Proof as in case II. In general, then Ap^λ is a residue of p^α when λ is greater than or equal to α and when λ is less than α and even and A a residue of p.

THE GENERAL SOLUTION OF A CLASS OF INFINITE SYSTEMS OF EQUATIONS IN AN INFINITE NUMBER OF VARIABLES.

By THOMAS. E. MASON, Indiana University.

The problem of this paper is to find the general solution of the system of equations

$$(1) \qquad \sum_{j=0}^{\infty} a_{ij}u_j = c_i, \quad i = 0,\ 1,\ 2,\ 3,\ \cdots,$$

where the coefficients a_{ij} are polynomials in i, of the form

$$a_{ij} = \alpha_j^{(0)} + \alpha_j^{(1)}i + \alpha_j^{(2)}i^2 + \cdots + \alpha_j^{(n)}i^n.$$

The system can be expressed in the form

$$(2) \qquad \sum_{j=0}^{\infty}\alpha_j^{(0)}u_j + i\sum_{j=0}^{\infty}\alpha_j^{(1)}u_j + i^2\sum_{j=0}^{\infty}\alpha_j^{(2)}u_j + \cdots + i^n\sum_{j=0}^{\infty}\alpha_j^{(n)}u_j = c_i, \qquad i = 0, 1, 2, \cdots.$$

A necessary and sufficient condition that the general solution of (2) shall be the general solution of (1) is that for every solution of (1) each of the infinite series indicated by the summations in (2) shall converge. This condition will be shown to be fulfilled and the system will be studied in the form (2).

When $i = 0$ the infinite series in the first term of (2) converges[1] since it is the first equation of (1). Now add the negative of that series to (1) and the result can be written in the form

$$(3) \qquad \sum_{j=0}^{\infty}(\alpha_j^{(1)}i + \alpha_j^{(2)}i^2 + \cdots + \alpha_j^{(n)}i^n)u_j = k_i, \quad i = 0, 1, 2, \cdots.$$

If any n consecutive equations of (3) are multiplied, the first by m_1, the second by m_2, $\cdots$, the nth by m_n, and the products added the result is of the form

$$\begin{aligned}\sum_{j=0}^{\infty}\{[m_1 i + m_2(i+1) + \cdots + m_n(i+n-1)]\alpha_j^{(1)} + [m_1 i^2 + m_2(i+1)^2 \\ + \cdots + m_n(i+n-1)^2]\alpha_j^{(2)} + \cdots + [m_1 i^n + m_2(i+1)^n \\ + \cdots + m_n(i+n-1)^n]\alpha_j^{(n)}\}u_j = k_i'.\end{aligned}$$

Now let t be any integer of the set $1, 2, 3, \cdots, n$. Then for each value of i the m's can be chosen in such way as to make the coefficient of $\alpha_j^{(t)}$ equal to 1 and the coefficients of $\alpha_j^{(k)}$ equal to 0, when $k \neq t$. That the m's can be so chosen is evident from the fact that they are thus restricted to satisfy n non-homogeneous equations in n unknowns, the determinant of the system being different from zero. This would give

$$\sum_{j=0}^{\infty}\alpha_j^{(t)}u_j = k_t'', \quad t = 1, 2, \cdots, n,$$

that is, each of the infinite series of (2) converges for every solution of (1). Hence the general solution of (2) is the general solution of (1) and vice versa.

Write

$$(4) \qquad \sum_{j=0}^{\infty}\alpha_j^{(t)}u_j = B_t, \quad t = 0, 1, 2, \cdots, n;$$

then (2) can be written

$$(5) \qquad B_0 + iB_1 + i^2B_2 + \cdots + i^nB_n = c_i, \quad i = 0, 1, 2, \cdots.$$

Difference this equation $n + 1$ times with respect to i and the result is

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[1] Only such values of u_j are called a solution as make every first member of (1) convergent.

$$\Delta^{n+1}c_i = 0, \tag{6}$$

that is,

$$c_{i+n+1} - (n+1)c_{i+n} + \frac{(n+1)n}{2!}c_{i+n-1} - \cdots + (-1)^{n+1}c_i = 0.$$

This is a necessary condition on the c's that equations (2) shall be consistent. The same condition on the c's can be obtained by taking consecutive values of i and eliminating the B's. We suppose that the c's satisfy this condition.

If $n+1$ consecutive values of i are taken in (5) the B's can be found in terms of the c's, since the determinant of the $n+1$ equations is different from zero. If these values of B_t are substituted in (4) the problem is reduced to solving the finite set of equations in an infinite number of variables:

$$\sum_{j=0}^{\infty} \alpha_j^{(t)} u_j = B_t, \quad t = 0, 1, 2, \cdots, n. \tag{7}$$

If possible select $n+1$ of the u's such that the determinant of their coefficients is different from zero. Transpose to the second member of the equations the remaining u's. These can be given arbitrary values, provided that the second member of each equation converges. Then the remaining $n+1$ u's can be found by the Cramer method. The solution thus obtained has an infinite number of arbitrary elements. Obviously it is the general solution.

If no $(n+1)$th order determinant, not zero, exists in the first member of (7) it will be shown below that the equations in the system (7) are linearly dependent. If the equations are linearly dependent and are consistent there will be conditions on the B's like

$$\sum_{t=0}^{n} b_t B_t = 0,$$

where the b's are constants. The number of such conditions will depend on the number of equations which are dependent on the others. Since the B's are expressed in terms of the c's these are additional conditions which must be fulfilled by the c's.

To show that if all the $(n+1)$th order determinants of the coefficients in (7) are zero the equations are linearly dependent, it will be shown that if they are linearly independent there is at least one $(n+1)$th order determinant different from zero.

Set up the matrix of the coefficients of (7)

$$\left\|\begin{array}{ccccc}
\alpha_0^{(0)} & \alpha_1^{(0)} & \alpha_2^{(0)} & \alpha_3^{(0)} & \cdots \\
\alpha_0^{(1)} & \alpha_1^{(1)} & \alpha_2^{(1)} & \alpha_3^{(1)} & \cdots \\
\alpha_0^{(2)} & \alpha_1^{(2)} & \alpha_2^{(2)} & \alpha_3^{(2)} & \cdots \\
\cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdots \\
\alpha_0^{(n-1)} & \alpha_1^{(n-1)} & \alpha_2^{(n-1)} & \alpha_3^{(n-1)} & \cdots \\
\alpha_0^{(n)} & \alpha_1^{(n)} & \alpha_2^{(n)} & \alpha_3^{(n)} & \cdots
\end{array}\right\|$$

The equations being assumed linearly independent, no row has each term zero, nor can any row be made to have each term zero by addition of multiples of other rows. If $\alpha_0^{(0)} = 0$ then we can move to the position of first column a column where the term in the first row is different from zero since not all the terms of any row are zero. Then by addition of multiples of this row all but the first term of the first column can be made zero. If $\alpha_1^{(1)} = 0$ then some column following has the term in the second row different from zero and it can be brought to occupy the position of second column. As before all the terms of this column below the second can be made zero by addition of multiples of the second row. Proceeding in this way a new matrix is obtained the first $(n + 1)$th order determinant of which will have all the terms below the principal diagonal equal to zero and each term of that diagonal different from zero. The determinant of the original matrix composed of those $n + 1$ columns which form the first $(n + 1)$th order determinant of the transformed matrix, is obviously not zero. Hence if the equations are linearly independent there will be at least one $(n + 1)$th order determinant different from zero, and hence if every $(n + 1)$th order determinant of the coefficients in (7) is zero the equations are linearly dependent.

In the case of linear dependence the equations which are dependent on others can be omitted and the general solution can be found from the linearly independent equations. This can be done as above by making arbitrary, with restriction as to convergence, all the u's except as many as there are equations and solving for these by the ordinary method.

PRECISE MEASUREMENTS WITH A STEEL TAPE OR WIRE.

By GEORGE R. DEAN, Rolla, Mo.

In the measurement of horizontal distances with a steel tape or wire in cases where all attainable accuracy is required, there will be considerable calculation avoided if the engineer has the means of knowing how to adjust the tension so that the elongation due to tension may balance the correction for sag or droop. Under ordinary conditions the correction for temperature is negligible, and when not negligible is easily applied and need not be discussed here.

We will derive a formula for the sag correction and the stretch separately. Then by equating these corrections, derive a formula for the tension in terms of the weight, cross-section and modulus of elasticity of the tape or wire.

Correction for Sag.—This is usually derived by the aid of the calculus in text-books on mechanics, but the calculus is unnecessary for this purpose, if we assume that the curve is the arc of a circle. The approximate formula derived in this way is the same as that obtained by integration when the curve is taken as a catenary or a parabola.